VII. *Mr.* Jones's *Computation of the Hyperbolic Logarithm of* 10 *improved: being a Transformation of the Series which he used in that Computation to others which converge by the Powers of* 80. *To which is added a Postscript, containing an Improvement of Mr.* Emerson's *Computation of the same Logarithm. By the Rev.* John Hellins, *Vicar of* Potter's Pury, *in* Northamptonshire. *Communicated by* Nevil Maskelyne, *D.D. F.R.S. and Astronomer Royal.*

Read February 18, 1796.

1. The method of computing by series is so extensive and useful a part of the mathematics, that any device which facilitates the operation by them will undoubtedly be acceptable to those who are proper judges of these matters. In this persuasion I have employed an hour of that little leisure which my present situation affords me, in improving a calculation of Napier's, for finding the hyperbolic logarithm of 10, which was given by the justly celebrated William Jones, Esq. F.R.S. in p. 180 of his *Synopsis Palmariorum Matheseos.* The same computation, described in a manner better suited to the capacities of beginners, was also published many years afterward by the learned Dr. Saunderson, in the second volume of his Elements of Algebra, p. 633 and 634. Since Dr. Saunderson's time the doctrine of series has been much improved. My present intention is, to exhibit a transformation of the series by which Mr. Jones computed the hyperbolic logarithm of 10

to others, the terms of which decrease by the powers of 80; so that their convergency is swift, and the divisions by 80 are easily made.

2. Mr. JONES considered the number 10 as composed of $2 \times 2 \times 2 \times \frac{5}{4}$; and consequently obtained the logarithm of 10 by adding three times the logarithm of 2 to the logarithm of $\frac{5}{4}$. The algebraic series which he used on this occasion was $\frac{2d}{s} + \frac{2d^3}{3s^3} + \frac{2d^5}{5s^5} + \frac{2d^7}{7s^7}$, &c. and the numerical value of $\frac{d}{s}$ was $\frac{1}{3}$ for the logarithm of 2, and $\frac{1}{9}$ for the logarithm of $\frac{5}{4}$; so that he has

$$\text{Sum of} \left\{ \begin{array}{l} 3\,\text{L.}\,2 = \frac{6}{3} + \frac{6}{3\cdot3^3} + \frac{6}{5\cdot3^5} + \frac{6}{7\cdot3^7}, \text{ \&c.} \\ \text{L.}\,\frac{5}{4} = \frac{2}{9} + \frac{2}{3\cdot9^3} + \frac{2}{5\cdot9^5} + \frac{2}{7\cdot9^7}, \text{ \&c.} \end{array} \right\} = \text{L. } 10.$$

3. Now the series $\frac{6}{3} + \frac{6}{3\cdot3^3} + \frac{6}{5\cdot3^5} + \frac{6}{7\cdot3^7}$, &c. $(= 3\text{L. } 2)$ is evidently $= \frac{6}{3} \times : 1 + \frac{1}{3\cdot3^2} + \frac{1}{5\cdot3^4} + \frac{1}{7\cdot3^6}$, &c. $= 2 \times : 1 + \frac{1}{3\cdot9} + \frac{1}{5\cdot9^2} + \frac{1}{7\cdot9^3}$, &c. And if the first, third, fifth, &c. term of this series be written in one line, and the second, fourth, sixth, &c. in another, we shall have

$$3\text{L. } 2 = \left\{ \begin{array}{l} \quad 2 \times : 1 + \frac{1}{5\cdot9^2} + \frac{1}{9\cdot9^4} + \frac{1}{13\cdot9^6}, \text{ \&c.} \\ + 2 \times : \frac{1}{3\cdot9} + \frac{1}{7\cdot9^3} + \frac{1}{11\cdot9^5} + \frac{1}{15\cdot9^7}, \text{ \&c.} \end{array} \right.$$

which two series are evidently

$$= \left\{ \begin{array}{l} \quad 2 \times : 1 + \frac{1}{5\cdot81} + \frac{1}{9\cdot81^2} + \frac{1}{13\cdot81^3}, \text{ \&c.} \\ + \frac{2}{9} \times : \frac{1}{3} + \frac{1}{7\cdot81} + \frac{1}{11\cdot81^2} + \frac{1}{15\cdot81^3}, \text{ \&c.} \end{array} \right.$$

And Mr. JONES's other series, $\frac{2}{9} + \frac{2}{3\cdot9^3} + \frac{2}{5\cdot9^5} + \frac{2}{7\cdot9^7}$, &c. $\left(=\text{L. } \frac{5}{4}\right)$ is evidently $= \frac{2}{9} \times : 1 + \frac{1}{3\cdot9^2} + \frac{1}{5\cdot9^4} + \frac{1}{7\cdot9^6}$, &c. $=$

$\frac{2}{9} \times : 1 + \frac{1}{3.81} + \frac{1}{5.81^2} + \frac{1}{7.81^3}$, &c. We therefore now have $3\,L.\,2 + L.\,\frac{5}{4}$ equal to the sum of these three series,

$$2 \times : 1 + \frac{1}{5.81} + \frac{1}{9.81^2} + \frac{1}{13.81^3}, \text{ \&c.}$$

$$\frac{2}{9} \times : \frac{1}{3} + \frac{1}{7.81} + \frac{1}{11.81^2} + \frac{1}{15.81^3}, \text{ \&c.}$$

$$\frac{2}{9} \times : 1 + \frac{1}{3.81} + \frac{1}{5.81^2} + \frac{1}{7.81^3}, \text{ \&c.}$$

which sum is also equal to the hyperbolic logarithm of 10.

4. The form to which Mr. JONES's series are now brought is evidently the same with the general form $a \times : \frac{1}{m} + \frac{x^n}{m+n} + \frac{x^{2n}}{m+2n} + \frac{x^{3n}}{m+3n}$, &c. the value of which, while m and n are affirmative numbers, and x sufficiently small, will be given by the series $a \times : \frac{1}{m.\overline{1-x^n}} - \frac{nx^n}{m.\overline{m+n}.\overline{1-x^n}|^2} + \frac{n.\,2n.\,x^{2n}}{m.\overline{m+n}.\overline{m+2n}.\overline{1-x^n}|^3} - \frac{n.\,2n.\,3n.\,x^{3n}}{m.\overline{m+n}.\overline{m+2n}.\overline{m+3n}.\overline{1-x^n}|^4}$,* &c. And this series, if we call the first, second, third, &c. terms of it A, B, C, &c. respectively, and put $\frac{x^n}{1-x^n} = z$, will be more concisely expressed thus; $a \times : \frac{1}{m.\overline{1-x^n}} - \frac{nzA}{m+n} + \frac{2nzB}{m+2n} - \frac{3nC}{m+3n} + \frac{4nzD}{m+4n}$, &c. which form is well adapted to arithmetical calculation.

Now, by comparing the three series at the end of the last article with the general series here given, we shall find that, in the first and last of these series, the value of m is 1, and in the second of them it is 3. The value of n in the first and

* See Phil. Trans. for 1794, Part 2d. p. 218, where this matter is more fully explained.

second series is 4, and in the third it is 2. The values of a are obviously 2 in the first series, and $\frac{2}{9}$ in the second and third. But in each of them z, $= \frac{x^n}{1 - x^n}$, is $= \frac{\frac{1}{81}}{1 - \frac{1}{81}} = \frac{1}{80}$. These values of the letters being written for them in the second general form, we have three new series, viz.

$$\frac{2.81}{80} - \frac{4A}{5.80} + \frac{8B}{9.80} - \frac{12C}{13.80} + \frac{16D}{17.80}, \text{ \&c.}$$

and $$\frac{2.81}{9.3.80} - \frac{4A}{7.80} + \frac{8B}{11.80} - \frac{12C}{15.80} + \frac{16D}{19.80}, \text{ \&c.}$$

and $$\frac{2.81}{9.80} - \frac{2A}{3.80} + \frac{4B}{5.80} - \frac{6C}{7.80} + \frac{8D}{9.80}, \text{ \&c.}$$

which three series are equal in value to those in art. 3, and to the hyperbolic logarithm of 10.

5. With respect to the convergency of these new series, it is evidently somewhat swifter than by the powers of 80. For even in the first series, which has the slowest convergency of the three, the coefficients $\frac{4}{5}$, $\frac{8}{9}$, $\frac{12}{13}$, &c. are each of them less than 1.

6. But another advantage of these new series is, that their numerators and denominators may be reduced to simpler terms, in consequence of which the arithmetical operation by them is further facilitated. In the first and second series, every term after the first is divisible by 4; and every term in the third series admits of a similar reduction by the number 2. The three series then, when these reductions are made, and their first terms are also abbreviated, will stand as below, (each still converging somewhat faster than by the powers of 80); and we shall have the hyperbolic logarithm of

$$=\begin{cases}\frac{81}{40}-\frac{A}{5.20}+\frac{2B}{9.20}-\frac{3C}{13.20}+\frac{4D}{17.20}, \text{ \&c.}\\ \frac{3}{40}-\frac{A}{7.20}+\frac{2B}{11.20}-\frac{3C}{15.20}+\frac{4D}{19.20}, \text{ \&c.}\\ \frac{9}{40}-\frac{A}{3.40}+\frac{2B}{5.40}-\frac{3C}{7.40}+\frac{4D}{9.40}, \text{ \&c.}\end{cases}$$

The arithmetical operation by the new series is undoubtedly easier than by the original series; yet it is evident, by inspection, that half the number of divisions by 20, (although easy operations), in the first and second series, may be exchanged for divisions by 10, which are no more than so many removals of the decimal point; and that, in the third series, half the number of divisions by 40, (the first excepted) may be exchanged for easier ones, one-fourth of them for divisions by 20, and the other fourth for divisions by 10. The new series then, still converging somewhat quicker than by the powers of 80, may stand thus:

$$\frac{81}{40}-\frac{A}{5.20}+\frac{B}{9.10}-\frac{3C}{13.20}+\frac{2D}{17.10}, \text{ \&c.}$$

and $\frac{3}{40}-\frac{A}{7.20}+\frac{B}{11.10}-\frac{3C}{15.20}+\frac{2D}{19.10}$, &c.

and $\frac{9}{40}-\frac{A}{3.40}+\frac{B}{5.20}-\frac{3C}{7.40}+\frac{D}{9.10}$, &c.

And even yet one might still facilitate the computation of the value of some of the terms. Thus, $\frac{3}{20}$ is $=\frac{1+\frac{1}{2}}{10}$; $\frac{3}{40}$ is $=\frac{1-\frac{1}{4}}{10}$; $\frac{5}{40}$ is $=\frac{1}{8}$; and $\frac{3}{15.20}$ is $=\frac{1}{100}$, &c.

By these expedients the sum of the three new series, which is equal to the hyperbolic logarithm of 10, may quickly be found.

P. S. *Containing an Improvement of Mr.* EMERSON's *Computation of the Hyperbolic Logarithm of* 10.

7. Since the above paper was written, on looking into EMERSON's Fluxions, I have found, at p. 137 of the first edition,* another computation of the hyperbolic logarithm of 10, which is preferable to Mr. JONES's, on account of the swifter convergency of one of the series used in it, as will appear presently. These series also admit of a transformation to others, by which the constant divisors 81 and 64009, used by Mr. EMERSON, are exchanged for 40 and 32000, while nearly the same rate of convergency is retained; which is another remarkable instance of the utility of transformations of this kind.

Mr. EMERSON, considering the number 10 as composed of $\frac{5^{10}\times 2^{30}}{4^{10}\times 10^{9}} = \frac{5^{10}}{4^{10}} \times \left.\frac{1024}{1000}\right|^{3}$, and using the same algebraic series as Mr. JONES used on this occasion, finds the hyperbolic logarithm of 10 to be $= 10$ L. of $\frac{5}{4} + 3$ L. of $\frac{1024}{1000}$,

$$= \begin{cases} \frac{20}{9} + \frac{20}{3.9.81} + \frac{20}{5.9.81^2} + \frac{20}{7.9.81^3}, \text{ \&c.} \\ + \frac{18}{253} + \frac{18.9}{3.253.64009} + \frac{18.9^2}{5.253.64009^2} + \frac{18.9^3}{7.253.64009^3}, \text{ \&c.} \end{cases}$$

where, instead of a series converging by the powers of $\frac{1}{9}$,† as in Mr. JONES's calculation, we have that which converges by the powers of $\frac{9}{64009}$, or above four times as swiftly. But what renders this very swiftly converging series still more useful is, that it admits of a transformation, by the theorem in article 4,

* See also page 197 of 3d edition.

† See article 3.

to another series which converges by the powers of $\frac{9}{64000}$, by which the numerical calculation is greatly facilitated.

8. For the two series in the preceding article (the sum of which is = H. L. of 10), are evidently =

$$\frac{20}{9} \times : 1 + \frac{1}{3.81} + \frac{1}{5.81^2} + \frac{1}{7.81^3}, \text{ \&c.}$$

and $\frac{18}{253} \times : 1 + \frac{9}{3.64009} + \frac{9^2}{5.64009^2} + \frac{9^3}{7.64009^3}$, &c.

And these two series, when transformed by the theorem abovementioned, and the terms abbreviated, become

$$\frac{9}{4} - \frac{A}{3.40} + \frac{2B}{5.40} - \frac{3C}{7.40} + \frac{4D}{9.40}, \text{ \&c.}$$

and $\frac{9.253}{32000} - \frac{9A}{3.32000} + \frac{2.9B}{5.32000} - \frac{3.9C}{7.32000} + \frac{4.9D}{9.32000}$, &c.

Which series admit of some other abbreviations similar to those pointed out in article *6*; and by them may the hyperbolic logarithm of 10 be very easily and expeditiously computed.

June 24, 1795.